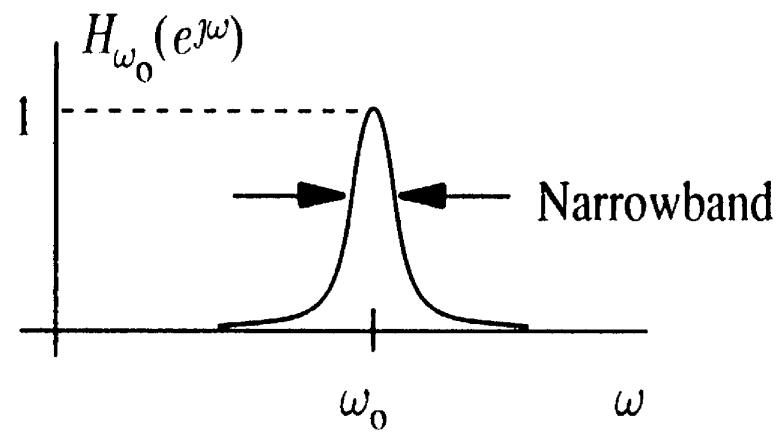
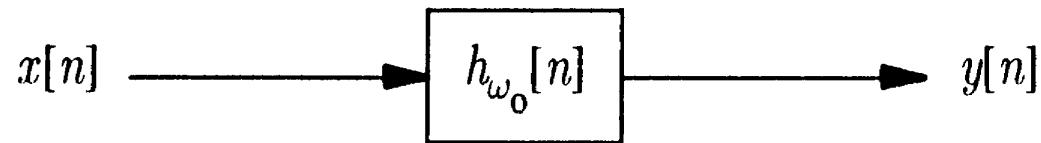


## **“MAXIMUM LIKELIHOOD” SPECTRUM ESTIMATION**

- Computed using the correlation matrix for the data
- Exhibits higher resolution than classical methods  
(Bartlett, Blackman-Tukey, etc.)
- Can be related to the “Maximum Entropy” method
- Also called the “Minimum Variance (Distortionless)” method

# “MAXIMUM LIKELIHOOD” METHOD

## SPECTRUM ANALYZER INTERPRETATION



ML spectral estimate:  $\hat{S}_{ML}(e^{j\omega_0}) \stackrel{\text{def}}{=} \mathcal{E}\{|y[n]|^2\}$

# ML METHOD DEVELOPMENT

The FIR filter output is

$$y[n] = \sum_{k=0}^{N-1} h_{\omega_0}[k]x[n-k] = \mathbf{h}_o^T \tilde{x}[n]$$

The filter average power is given by

$$\mathcal{P} = E\{|y[n]|^2\} = \mathbf{h}_o^T E\{\tilde{x}[n]\tilde{x}^{*T}[n]\} \mathbf{h}_o^* = \mathbf{h}_o^T \tilde{\mathbf{R}}_x \mathbf{h}_o^* = \mathbf{h}_o^{*T} \mathbf{R}_x \mathbf{h}_o$$

This is minimized subject to the (complex) constraint

$$H_{\omega_0}(e^{j\omega_0}) = \sum_{n=0}^{N-1} h_{\omega_0}[n]e^{-j\omega_0 n} = \mathbf{w}_o^{*T} \mathbf{h}_o = 1$$

## ML METHOD (cont'd.)

The optimization problem involves the Lagrangian

$$\mathcal{L} = \mathbf{h}_o^{*T} \mathbf{R}_x \mathbf{h}_o + \mu(1 - \mathbf{w}_o^{*T} \mathbf{h}_o) + \mu^*(1 - \mathbf{h}_o^{*T} \mathbf{w}_o)$$

and the necessary condition

$$\nabla_{\mathbf{h}_o^*} \mathcal{L} = \mathbf{R}_x \mathbf{h}_o - \mu^* \mathbf{w}_o = 0 \quad \implies \quad \mathbf{h}_o = \mu^* \mathbf{R}_x^{-1} \mathbf{w}_o$$

The requirement  $\mathbf{w}_o^{*T} \mathbf{h}_o = \mu^* \mathbf{w}_o^{*T} \mathbf{R}_x^{-1} \mathbf{w}_o = 1$  then yields

$$\mu^* = \mu = \frac{1}{\mathbf{w}_o^{*T} \mathbf{R}_x^{-1} \mathbf{w}_o} \quad \text{so that} \quad \mathbf{h}_o = \frac{\mathbf{R}_x^{-1} \mathbf{w}_o}{\mathbf{w}_o^{*T} \mathbf{R}_x^{-1} \mathbf{w}_o}$$

## ML METHOD (cont'd.)

The optimum narrowband filter at frequency  $\omega_0$

$$h_o = \frac{\mathbf{R}_x^{-1} w_o}{w_o^{*T} \mathbf{R}_x^{-1} w_o}$$

produces the output power

$$\mathcal{P} = h_o^{*T} \mathbf{R}_x h_o = \frac{w_o^{*T} \mathbf{R}_x^{-1} \mathbf{R}_x \mathbf{R}_x^{-1} w_o}{(w_o^{*T} \mathbf{R}_x^{-1} w_o)^2} = \frac{1}{w_o^{*T} \mathbf{R}_x^{-1} w_o}$$

This is the ML power spectral estimate of the process at frequency  $\omega_0$ .

## “MAXIMUM LIKELIHOOD” METHOD (SUMMARY)

The “Maximum Likelihood” spectral estimate is

$$\hat{S}_{ML}(e^{j\omega}) = \frac{1}{\mathbf{w}^* \mathbf{R}_x^{-1} \mathbf{w}}$$

where

$$\mathbf{w} = \begin{bmatrix} 1 \\ e^{j\omega} \\ e^{j2\omega} \\ \vdots \\ e^{j(N-1)\omega} \end{bmatrix}$$

## CLASSICAL METHOD COMPARED TO “MAXIMUM LIKELIHOOD” METHOD

If the Fourier transform of the data sequence is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \mathbf{w}^{*T} \mathbf{x}$$

the periodogram spectral estimate is defined by

$$\hat{P}_x(e^{j\omega}) \stackrel{\text{def}}{=} \frac{1}{N} |X(e^{j\omega})|^2 = \frac{1}{N} X(e^{j\omega}) X^*(e^{j\omega}) = \frac{1}{N} \mathbf{w}^{*T} \mathbf{x} \mathbf{x}^{*T} \mathbf{w}$$

The expected value of this estimate is

$$E \left\{ \hat{P}_x(e^{j\omega}) \right\} = \frac{1}{N} \mathbf{w}^{*T} \mathbf{R}_{\mathbf{x}} \mathbf{w} \quad \text{while} \quad \hat{S}_{ML}(e^{j\omega}) = \frac{1}{\mathbf{w}^{*T} \mathbf{R}_{\mathbf{x}}^{-1} \mathbf{w}}$$

## RELATION BETWEEN ML AND ME

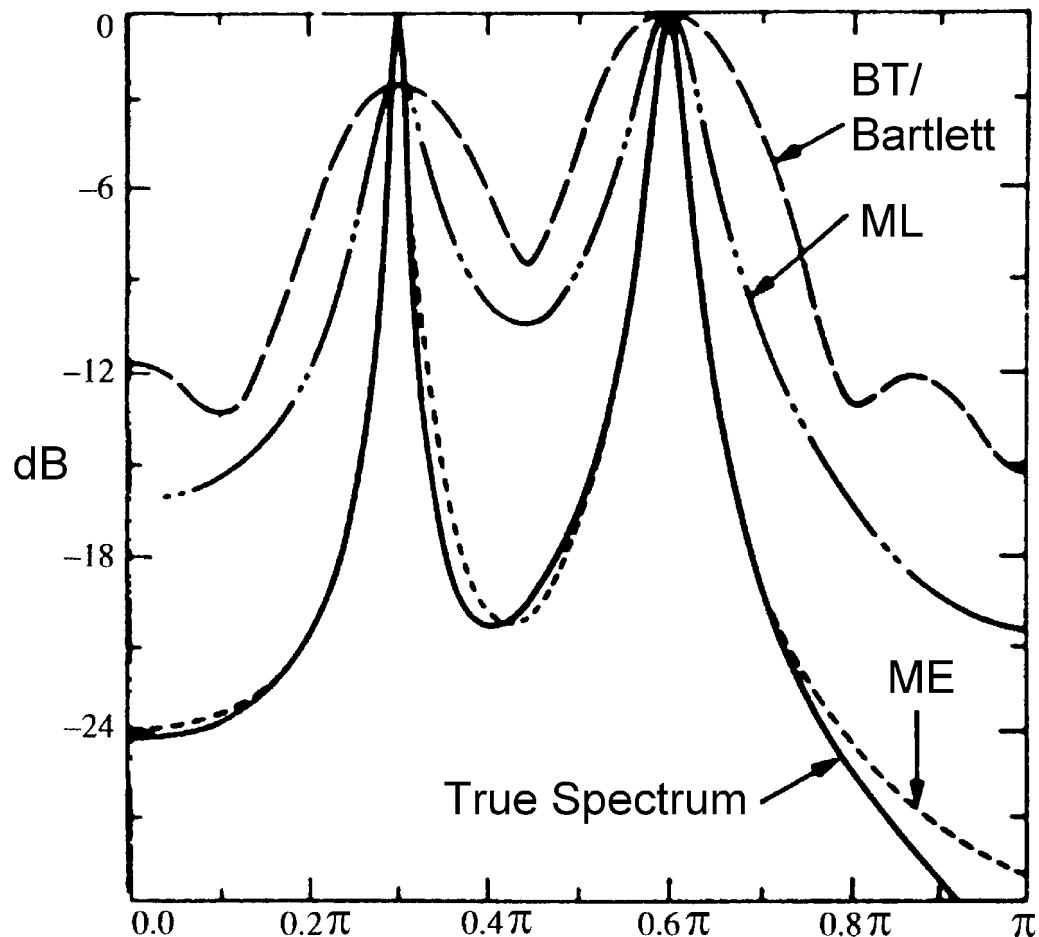
Use the triangular decomposition to write

$$\begin{aligned}
 \hat{S}_{ML}^{-1}(e^{j\omega}) &= \mathbf{w}^{*T} \mathbf{R}_x^{-1} \mathbf{w} = \mathbf{w}^{*T} (\mathbf{U}_1^{-1})^{*T} \mathbf{D}_U^{-1} \mathbf{U}_1^{-1} \mathbf{w} \\
 &= \mathbf{w}^{*T} \begin{bmatrix} 1 & \cdots & 0 & 0 \\ a_1^{(N-1)} & \ddots & \vdots & \vdots \\ \vdots & \vdots & 1 & 0 \\ a_{N-1}^{(N-1)} & \cdots & a_1^{(1)} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_{N-1}^2} & 0 & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \frac{1}{\sigma_1^2} & 0 \\ 0 & \cdots & 0 & \frac{1}{\sigma_0^2} \end{bmatrix} \begin{bmatrix} 1 & \cdots & \cdots & a_{N-1}^{(N-1)*} \\ \vdots & \ddots & \cdots & \cdots \\ 0 & \cdots & 1 & a_1^{(1)*} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{w} \\
 &= \sum_{p=0}^{N-1} \frac{\left| \sum_{k=0}^p a_k^{(p)} e^{-j\omega k} \right|^2}{\sigma_p^2}
 \end{aligned}$$

Then ...

$$\frac{1}{\hat{S}_{ML}(e^{j\omega})} = \sum_{p=0}^{N-1} \frac{1}{\hat{S}_{ME}^{(p)}(e^{j\omega})}$$

# SPECTRUM ANALYSIS: COMPARISON



$$R_x[l] = e^{-0.02l} (\cos 0.3\pi l + \frac{1}{15\pi} \sin 0.3\pi l)$$
$$+ 2e^{-0.04l} (\cos 0.6\pi l + \frac{1}{15\pi} \sin 0.6\pi l)$$

(11 samples)

# COMPUTATION OF THE ML ESTIMATE

## FASTEST METHOD:

1. Express the denominator as

$$\mathbf{w}^{*T} \hat{\mathbf{R}}_{\mathbf{x}}^{-1} \mathbf{w} = \sum_{k=-N}^N \varrho[k] e^{-j\omega k}$$

where  $\varrho[k]$  is the sum of terms on diagonals of  $\hat{\mathbf{R}}_{\mathbf{x}}^{-1}$

2. Use the FFT to compute this term and take reciprocal